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Non-linear-Maxwell-element-type hysteretic control force

Kazuhiko Yamada*

Kobori Research Complex, Kajima Corporation, KI Building, 6-5-30 Akasaka, Minato-ku, Tokyo 107-8502, Japan

SUMMARY

A simple non-linear control law is proposed for reducing structural responses against seismic excitations. This law defines control force dynamics by one differential equation involving a non-linear term that restrains the control force amplitude. If non-linearity is neglected, the control force becomes the force in a Maxwell element, so it is called the non-linear-Maxwell-element-type (NMW) control force. The NMW control force vs. deformation relation plots hysteretic curves. The basic performance of an SDOF model with the NMW control force is examined for various conditions by numerical analyses. Furthermore, the control law is extended to fit an MDOF structural model, and an application example is shown. The computational results show that the NMW control force efficiently reduces structural responses. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: structural control; non-linear control; Maxwell-element; hysteresis; seismic response; damping

INTRODUCTION

The most basic active control systems using a velocity feedback law or the least quadratic regulator theory reduce structural responses by damping effects. The control effect is essentially equivalent to that of damping devices used in passive systems, although active systems can reduce responses more. According to the linear control theory, the greater the control effect, the larger the control system. However, a practical control system must satisfy various economical and mechanical constraints.

Non-linear control laws can in fact satisfy various constraints. A direct way of introducing a non-linear control law is to set a non-linear optimization problem, formulating constraints and a cost function [1]. Other non-linear control laws are also introduced [2, 3] by setting a cost function considering higher order norms. However, they require heavy computation.

By contrast, this paper proposes a simple control law that restrains the control force amplitude, thus satisfying economical and mechanical constraints. One non-linear differential equation expresses it, which is the so-called indirect control method [4]. It defines the control force dynamics with an in-advance non-linear term for the control force and the relative velocity where it acts. If non-linearity is neglected, the control force becomes the force in a Maxwell element.

* Correspondence to: Kazuhiko Yamada, Kobori Research Complex, Kajima Corporation, KI Building, 6-5-30 Akasaka, Minato-ku, Tokyo 107-8502, Japan

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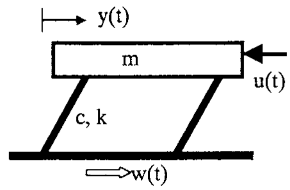

Figure 1. SDOF model.

Thus, the control force produced by the proposed law is called the non-linear-Maxwell-element-type (NMW) control force. Because the proposed control law considers a simple non-linear term of the control force and non-linear influence can be independently determined for every control force, it can be easily applied to multi-degree of freedom (MDOF) model. The proposed control law is not introduced by complex calculation for optimization, but is based on a mathematical view of non-linear dynamics for an elastoviscoplastic constitutive law [5,6].

Thus, the control force vs. deformation relation plots as hysteretic curves as stress vs. strain relations in a structural element that yields under a large load. While a control law can be based on a Bouc–Wen hysteretic model [7], the author deduces only necessary terms that show the anticipated behaviour. In fact, the proposed control law, which is applicable to control systems directly supplying control forces, is another version of the nonlinear control law for a variable damper defined by differential equation [8].

The following first introduces the control law for a single-degree-of-freedom (SDOF) model and discusses its stability. Basic performances of an SDOF model for impulse, sinusoidal and seismic excitations are examined by numerical analyses. Next, the proposed control law is extended for an MDOF structural model. Parameters in the MDOF structural model are related to those in an SDOF model by mode decomposition. Numerical analysis using a 3DOF structural model shows that the proposed law more efficiently reduces structural responses than velocity feedback laws.

CONTROL LAW FOR SDOF MODEL

Equation of motion: Let us consider an SDOF model of a structure with a control force under a seismic excitation (see Figure 1). Then, the instantaneous momentum balance at time t is

$$my''(t) + cy'(t) + ky(t) = -mw(t) + u(t) \tag{1}$$

where $y''(t)$, $y'(t)$ and $y(t)$ represent response acceleration, velocity and displacement, respectively, m, c, k are structural mass, damping and stiffness, respectively, $u(t)$ is a control force, and $w(t)$ is acceleration at the structural basement of the seismic excitation.

Control force: Now, let the control force satisfy the control law:

$$u'(t) = -gy'(t) - e_{\mathrm{L}}u(t) - e_{\mathrm{N}}u(t)^3 \tag{2}$$

where g, e_{L} and e_{N} are feedback gains and are positive. If $e_{\mathrm{N}} = 0$, the control force is equivalent to the force produced in a Maxwell element. Thus, the control force defined by Equation (2) is called the non-linear-Maxwell-element type (NMW).

When the control force becomes large, the non-linear term, i.e. the third term on the right-hand side, dominates and restrains the control force increment. With a small control force increment, it is approximately proportional to the cube root of the velocity with coefficient $(g/e_{\mathrm{N}})^{1/3}$. This control influence is called the damping effect. However, the value of a non-linear term, i.e. the third term on the right-hand side of Equation (2), becomes small when the control force is small. The change rate of control force is thus almost proportional to the velocity, that is, the control force is almost proportional to deformation. The control force then works like a stiffness element with linear damping coefficient g/e_{L}. This control influence is called stiffness effect. The third power of the control force effectively adjusts stiffness and damping effects to reduce responses, and restrains the control force from becoming large. Although the non-linear term can be other odd functions of the control force, the third power is chosen here for simplicity. As a result, the slope of the NMW control force vs. deformation is steep for small deformations, while it flattens for large deformations. The NMW control force vs. deformation relation plots parallelogram-like hysteretic curves by stiffness and damping effect, while the Maxwell-element-type (MW) control force plots elliptic curves. These parallelogram-like hysteretic curves are similar to those for an elastoplastic element. The proposed law in fact resembles an elastoviscoplastic constitutive law, so the NMW control force acts as if an elastoviscoplastic element were installed.

Stability under impulse excitation: The free vibration is asymptotically stable when controlled by the proposed law. The following is a proof.

Proof. Let a Lyapunov function candidate V be

$$V(y, y', u) = (my'^2/2) + (ky^2/2) + (u^2/2g) \tag{3}$$

Then

$$\begin{aligned} V'(y, y', u) &= my'y'' + kyy' + uu'/g = y'(my'' + ky) + u(-gy' - e_{\mathrm{L}}u - e_{\mathrm{N}}u^3)/g \\ &= y'(-cy' + u) - uy' - e_{\mathrm{L}}u^2/g - e_{\mathrm{N}}u^4/g = -cy'^2 - (e_{\mathrm{L}} + e_{\mathrm{N}}u^2)u^2/g \end{aligned} \tag{4}$$

Hence, (i) $V(y,y',u) = 0$ only if $(y, y', u) = (0, 0, 0)$; (ii) $V(y,y',u) > 0$, $V'(y,y',u) < 0$ unless $(y, y', u) = (0, 0, 0)$. Therefore, $(y, y', u) = (0, 0, 0)$ is asymptotically stable.

Energy balance: By multiplying Equation (1) by $y'(t)$ and Equation (2) *by* $u(t)$, and integrating the result from t_1 to t_2, the energy balance equation is obtained:

$$\begin{aligned} &[\{my'(t_2)^2/2 + ky(t_2)^2/2\} - \{my'(t_1)^2/2 + ky(t_1)^2/2\}] + \left[\int_{t_1}^{t_2} cy'(t)^2 \mathrm{d}t\right] \\ &+ \left[\{u(t_2)^2 - u(t_1)^2\}/(2g) + (1/g)\int_{t_1}^{t_2} (e_{\mathrm{L}} + e_{\mathrm{N}}u(t)^2)u(t)^2\,\mathrm{d}t\right] = \left[-\int_{t_1}^{t_2} my'(t)w(t)\mathrm{d}t\right] \end{aligned} \tag{5}$$

Let us call the four terms in brackets the vibration, damping, control force and input energies, respectively. The term $(1/g)\int_{t_1}^{t_2}(e_{\mathrm{L}} + e_{\mathrm{N}}u(t)^2)u(t)^2\,\mathrm{d}t$ for the control force indicates energy dissipation from t_1 to t_2 by the control force. On condition that the initial and end states are still, the vibration energy and $\{u(t_2)^2 - u(t_1)^2\}/(2g)$ are zero. Structural damping and the control force dissipate the input energy, provided e_{L}, e_{N} and g are positive. This implies that the control law is robust for parameter fluctuation.

Scaling: We can introduce scaled equations for Equations (1) and (2) by letting $y = Yy^*$, $\tau = \omega t$, $\omega^2 = k/m$ and $2h\omega = c/m$,

$$\ddot{y}^*(\tau) + 2h\dot{y}^*(\tau) + y^*(\tau) = -w^*(\tau) + u^*(\tau) \tag{6}$$

$$\dot{u}^*(\tau) = -g^*\dot{y}(\tau) - e_{\mathrm{L}}^* u(\tau) - e_{\mathrm{N}}^* u^*(\tau)^3 \tag{7}$$

where $w^*(\tau) = w(t)/Y\omega^2$, $u^*(\tau) = u(t)/m\omega^2 Y$, $g^* = g/m\omega^2$, $e_{\mathrm{L}}^* = e_{\mathrm{L}}/\omega$, $e_{\mathrm{N}}^* = e_{\mathrm{N}} m^2 \omega^3 Y^2$, $\partial y/\partial\tau = \dot{y}$, $\partial^2 y/\partial\tau^2 = \ddot{y}$ and $\partial u^*/\partial\tau = \dot{u}^*$. Thus, models possessing the same normalized parameters should show similar control force influence. That is, to plot a geometrically similar control force vs. displacement relation to the original for the case of 10-times the mass, twice the natural frequency and three times the level input, g, e_{L} and e_{N} must be multiplied by 10×2^2, 2 and $10^{-2} \times 2^{-3} \times 3^{-2}$, respectively.

g^* should be 1.0 or greater, but to avoid undesirable high-frequency vibration it should not be too large. We can estimate appropriate e_{L}^* and e_{N}^*, considering the studies described in a later section and the fact that g^*/e_{L}^* is a damping coefficient of a Maxwell element.

Control delay: By allowing a small control delay θ, a more practical control law is introduced as

$$u(t) = u(t-\theta) - \theta\{gy'(t-\theta) - e_{\mathrm{L}}u(t-\theta) - e_{\mathrm{N}}u(t-\theta)^3\} \tag{8}$$

The larger the g, e_{N} and input level, which induce strong non-linearity, the smaller the control delay.

Then, we can compute the basic performances by numerical analyses, assuming computational time interval Δt and using the discrete form of Equation (1) at time t.

BASIC PERFORMANCE OF NMW CONTROL FORCE

Let us examine the basic performance of the NMW control force, considering an SDOF model with $m = 1.0$, $c = 0.04\pi$ and $k = 4\pi^2$. The assumed natural frequency and damping factor of the model are 1.0 Hz and 0.01, respectively. The units for the assumed parameters other than time are unspecified.

For impulse: Assuming that the initial velocities are 0.1, 0.5 and 1.0, free vibration responses are computed for the SDOF model with the NMW, MW and simple velocity feedback (VF) control force. Three cases, case-NMW, case-MW and case-VF are computed with $\{g, e_{\mathrm{L}}, e_{\mathrm{N}}\} = \{4\pi^2, 5, 50\}$, $\{4\pi^2, 28, 0\}$, $\{\text{infinity}, 0.2\pi, 0\}$, respectively. Control delay $\theta = 0.01$ s is assumed for all cases.

The resulting control force vs. displacement (u–y) relations and displacement time histories are shown in Figures 2 and 3. All assumed cases do not show undesirable influence from the assumed control delay, so that $\theta = 0.01$ s is allowable. For $y'(0) = 0.1$, a large NMW control force acts on a model quickly, and the natural frequency of the model becomes short due to stiffness effect. Then, not only is the maximum displacements about 85% of the others, but also the structural responses are immediately reduced. The difference between case-MW and case-VF is small, so that g is large enough to induce a damping effect. For $y'(0) = 0.5$, the maximum control forces are almost the same. However, the NMW control force mostly maintains its peaks for a while, thus distorting the MW control force vs. displacement relation. Thus, the NMW control force reduces responses a little more quickly than others. For $y'(0) = 1.0$, the maximum NMW control force is

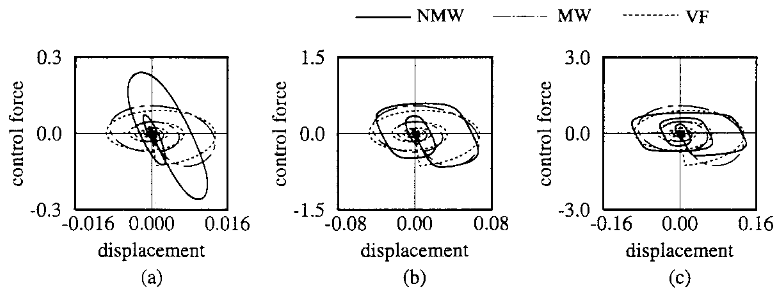

Figure 2. Control force vs. displacement relations for impulse with: (a) $y'(0) = 0.1$; (b) $y'(0) = 0.5$; (c) $y'(0) = 1.0$.

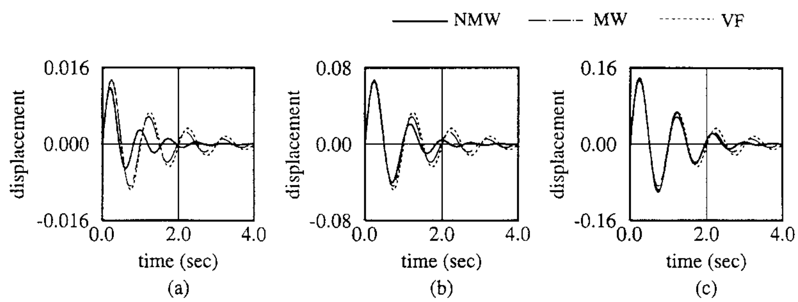

Figure 3. Displacement time histories for impulse with: (a) $y'(0) = 0.1$; (b) $y'(0) = 0.5$; (c) $y'(0) = 1.0$.

about 70% those of others, but still reduces responses more quickly than others. Therefore, the NMW control force can efficiently reduce structural responses.

For sine excitations: Next, let us assume case-NMW when excited by sine waves of various frequencies, producing a resonance curve. The excitation frequencies are 0.3–2.0 Hz and their maximum accelerations are 0.1, 0.5 and 1.0. The excitation levels are called S-0.1, S-0.5 and S-1.0 after the assumed maximum acceleration of input waves. Figure 4 shows the results.

For S-0.1, the NMW control force greatly reduces the resonance curve peak and shifts the model's natural frequency to near 1.4 times the original. For S-0.5, frequency shift and peak reduction are smaller. For S-1.0, both stiffness and damping effects are small. That is, the NMW control force not only reduces the resonant peak by a damping effect, but also shifts the equivalent natural frequency of the model by a stiffness effect. The assumed excitation levels influence these effects as well as the control force amplitude.

For seismic excitations: Next, let us excite a model controlled by the NMW control force by seismic excitations, scaled El Centro 1940 NS (El Centro). Their maximum accelerations are assumed to be 0.5–7.0, and they are called e.g., E-0.5 after their maximums. Furthermore, the control effects are examined by changing e_N to 25 and 100 as well as 50. Figure 5 shows the resulting u–y relations for E-1.0, E-3.5 and E-7.0, and Figure 6 shows the maximum displacement and control force for various levels.

The u–y relations plot complex curves when subjected to seismic excitations. The large area surrounded by the curve implies high damping effect, while a high-slope straight line implies

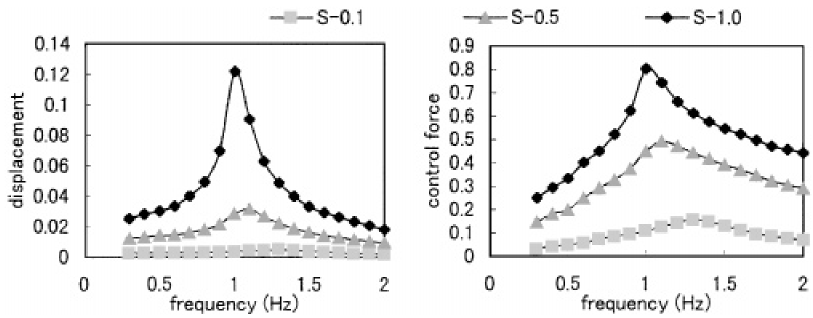

Figure 4. Maximum displacements and control forces for various-frequency sine excitations.

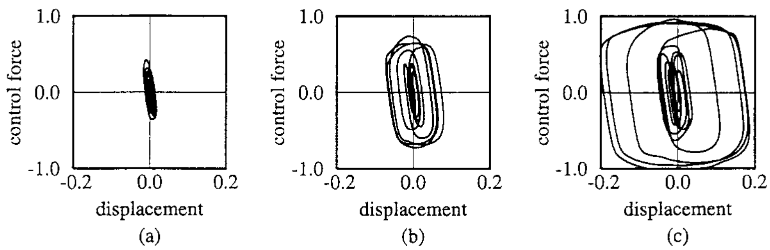

Figure 5. Control force vs. displacement relations for: (a) E-1.0; (b) E-3.5; (c) E-7.0.

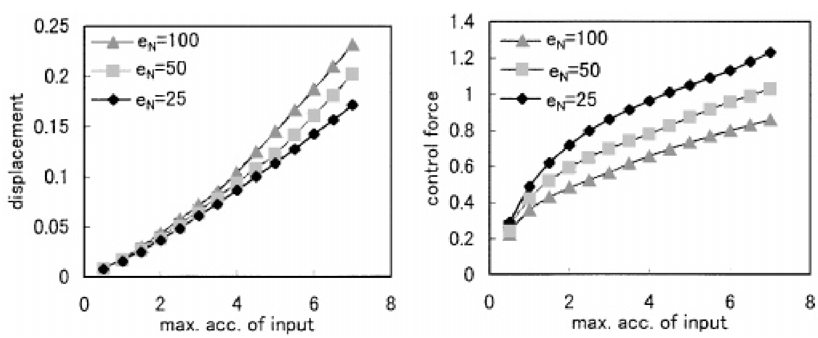

Figure 6. Maximum displacement and control force for various-level El Centro.

a strong stiffness effect. Stiffness effect is strong for E-1.0, while damping effect is high for E-7.0. The control force amplitude does not increase in proportion to input levels. The maximum control force for E-7.0 is about twice of that for E-1.0 and 1.2 times of that for E-3.5. Because the control force increment is restrained, the maximum displacement for E-7.0 is more than twice of that for E-3.5. Figure 6 more clearly shows these tendencies and that the non-linearity influence increases with e_N values. That is, large e_N restrains the control force amplitude for large-level excitations, thus reducing the displacement less.

DYNAMICS OF MDOF MODEL

MDOF model: Next, let us consider an MDOF structural model, n-dimensional vectors $y''(t)$, $y'(t)$ and $y(t)$ represent response acceleration, velocity and displacement, respectively, of an nDOF structural model at time t relative to the structural basement. Let $u(t)$ represent a control force and be an m-dimension vector, so multiple control forces can be operated. Furthermore, let $w(t)$ be acceleration at the structural basement excited by a seismic motion. These are expressed in the same notation as those in Equation (1). Then, structural motion is expressed by

$$My''(t) + Cy'(t) + Ky(t) = -MVw(t) + Uu(t) \tag{9}$$

where M, C and K are $(n \times n)$-dimensional matrices representing model's mass, damping and stiffness, respectively. V is an n-dimensional vector that indicates the DOF where the seismic excitation acts. U is an $(n \times m)$-dimensional matrix which represents the DOF where the control forces act.

NMW control force: Let the law for the NMW control force be defined by

$$u'(t) = -GU^{\mathrm{T}}y'(t) - E_{\mathrm{L}}u(t) - E_{\mathrm{N}}u^3(t) \tag{10}$$

where $u^3(t) = \{u_1(t)^3, \ldots, u_j(t)^3, \ldots, u_m(t)^3\}^{\mathrm{T}}$, i.e. $u^3(t)$ indicates a vector whose components are the third power of each control force and $u'(t)$ is the control force rate at time t, i.e. time derivative of the control force. G, E_{L} and E_{N} represent feedback gains, which are given by

$$G = \mathrm{diag}\{g_j\}, \quad E_{\mathrm{L}} = \mathrm{diag}\{e_{\mathrm{L}_j}\}, \quad E_{\mathrm{N}} = \mathrm{diag}\{e_{\mathrm{N}_j}\} \tag{11}$$

where diag$\{\ \}$ composes a diagonal matrix of a vector $\{\ \}$. Thus, the control force dynamics depend on the deformation rate between the DOFs and the amount of the control force at time t.

Mode decomposition: Let ω_i and ϕ_i be the ith natural frequency and normalized mode vector, respectively, i.e. $\omega_i^2 M\phi_i = K\phi_i$ and $\phi_i^{\mathrm{T}}\phi_i = 1$. Then, Equations (9) and (10) are converted to

$$\phi_i^{\mathrm{T}}M\phi_i\phi_i^{\mathrm{T}}y''(t) + \phi_i^{\mathrm{T}}C\phi_i\phi_i^{\mathrm{T}}y'(t) + \phi_i^{\mathrm{T}}K\phi_i\phi_i^{\mathrm{T}}y(t) = -\phi_i^{\mathrm{T}}MVw(t) + \phi_i^{\mathrm{T}}Uu(t) \tag{12}$$

$$u'(t) = -GU^{\mathrm{T}}\phi_i\phi_i^{\mathrm{T}}y'(t) - E_{\mathrm{L}}u(t) - E_{\mathrm{N}}u^3(t). \tag{13}$$

Then, by assuming that the damping matrix can be decomposed as $2h_i\omega_i = \phi_i^{\mathrm{T}}M^{-1}C\phi_i$, and letting u_j and U_j be the jth component of u and U, respectively, the ith mode and the jth control force are expressed as

$$\tilde{y}_i''(t) + 2h_i\omega_i\tilde{y}_i'(t) + \omega_i^2\tilde{y}_i(t) = -\beta_i w(t) + \textstyle\sum_j \alpha_{ij}u_j(t) \tag{14}$$

$$u_j' = -\textstyle\sum_i \tilde{g}_{ji}\tilde{y}_i'(t) - e_{\mathrm{L}_j}u(t) - e_{\mathrm{N}_j}u_j(t)^3 \tag{15}$$

where $\tilde{y}_i(t) = \phi_i^{\mathrm{T}}y(t)$, $\beta_i = \phi_i^{\mathrm{T}}V$, $\alpha_{ij} = \phi_i^{\mathrm{T}}U_j$, and $\tilde{g}_{ji} = g_jU_j^{\mathrm{T}}\phi_i$.

It should be remarked that the mode vectors do not decompose $e_{\mathrm{N}j}$, that is, the nonlinear influence is independent of the mode vectors. Thus, parameters for an MDOF model can be determined as in an SDOF model through mode decomposition.

APPLICATION TO 3DOF STRUCTURE

Let us consider a three-storey structure with control systems and model it as a 3DOF system, taking account of the weights at each floor and springs among storeys, as shown in Figure 7. The

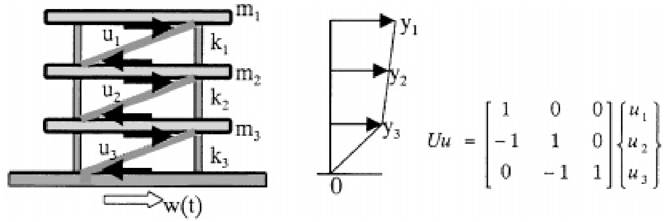

Figure 7. MDOF Model.

assumed weights and spring stiffnesses from the top to the bottom are:

weights: 1960, 1176, 1176 (kN); spring stiffnesses: 19 600, 19 600, 24 500 (kN/m).

Thus, spring forces mean structural shear forces between each storey. Then, the first natural period of the model is about 1.23 s. An internal viscous damping with a factor $h = 0.05$ for 1.0 s is assumed for all springs. These control systems can produce control forces that act laterally between floors. The assumed seismic excitation is El Centro whose maximum accelerations are scaled to 0.35, 3.5 and 7.0 $\mathrm{m/s^2}$, which are called E-0.35, E-3.5 and E-7.0, respectively. Three cases, case-NMW, case-VF and case-NO, are computed. For case-NMW, the NMW control force is assumed. Then, the parameters from the top to the bottom are $\{g_j\} = \{-19\,600,\ -19\,600,\ -24\,500\}$ (kN/m); $\{e_{\mathrm{N}j}\} = \{-0.00125,\ -0.00125,\ -0.00125\}$ (1/MN^{2}s). For case-VF, velocity feedback control forces are assumed. The gains to the deformation rate from the top to the bottom are $\{-620,\ -620,\ -780\}$ (kNs/m), which means 10 per cent additional damping. For case-NO, no control force is assumed.

Figure 8 shows deformation, deformation rate and control force time histories for case-NMW at the second storey for E-7.0. The sign of the control force is reversed, so that it can be easily compared with deformation and deformation rate. From 1.5 to 7.0 s, where the amplitudes are large, the control force and the deformation rate have almost the same phase with about 1.2-s period, which means less change from the original. Then, the control force sharply changes from one peak to another and is flat near its peaks. However, the phase of the control force corresponds to that of the deformation during other times where the amplitude is small. Thus, the responses are efficiently reduced, by the damping and stiffness effects.

Figure 9 shows that u–y relations at the second storey for case-NMW for E-0.35, E-3.5 and E-7.0. For E-0.35, the u–y relation is elliptic; thus energy is dissipated. The maximum control force does not proportionally increase with input levels. For E-3.5 and E-7.0, the u–y relation plots parallelogram-like curves, so that the control force almost keeps its amplitude from one deformation peak to another while it quickly changes its direction near deformation peaks.

Figure 10 shows the maximum control forces and spring forces at each storey. For E-0.35, the control forces for case-NMW are more than twice of those for case-VF. The spring forces then are about half of those of case-NO. Thus, the large and effective control forces reduce structural responses for case-NMW. For E-3.5, not only the control forces but also the spring forces are almost the same for case-NMW and case-VF. Thus, the effects of the NMW control forces for E-3.5 are evaluated as almost 10 per cent additional damping. For E-7.0, the control forces for case-NMW are only 7.5 per cent of the spring forces for case-NO, and $\frac{2}{3}$ of those for case-VF. However, they can reduce the spring forces by about $\frac{1}{3}$ for case-NO, which does not much differ

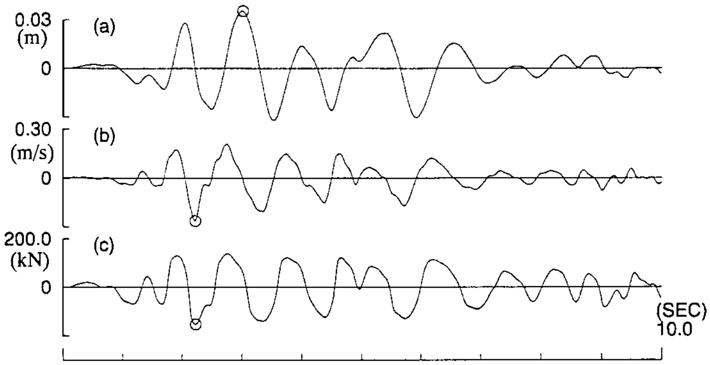

Figure 8. Response time histories for E-3.5: (a) deformation; (b) deformation rate; (c) -1.0* control force.

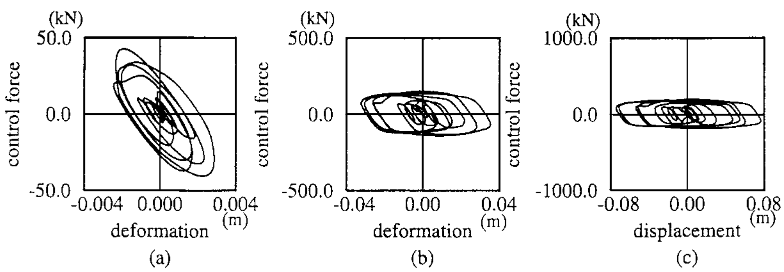

Figure 9. Control force vs. deformation relation at the second storey for: (a) E-0.35; (b) E-3.5; (c) E-7.0.

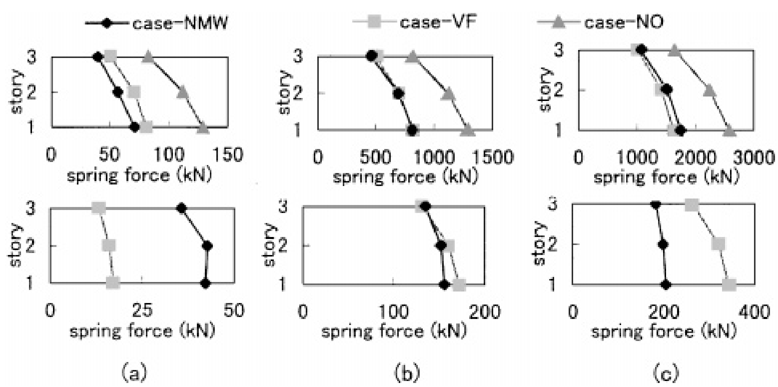

Figure 10. Maximum spring forces and control forces for: (a) E-0.35; (b) E-3.5; (c) E-7.0.

from those for case-VF. Therefore, the NMW control forces efficiently reduce structural responses.

To understand the basic performance, the parameters in the foregoing cases are determined to show almost equivalent control forces for a middle-level excitation. However, those in practical

cases should be determined for the largest level. In that case, the NMW control force can use its full ability to suppress responses to smaller excitations. Thus, we can design a compact control system which efficiently reduces structural responses.

CONCLUSIONS

(1) A non-linear active control law has been proposed for reducing structural responses to seismic excitations. This control law is expressed by one differential equation involving a non-linear term, a cubic term of the control force or its discrete form considering control delay. The resulting control force is called the non-linear-Maxwell-element type (NMW). The NMW control force vs. deformation relation plots a parallelogram-like hysteretic curve, dissipating energy and restraining its amplitude.
(2) The performance of the SDOF model according to the proposed control law was examined by numerical analyses. The results showed that the NMW control force more efficiently reduces structural responses than a velocity feedback control law.
(3) The proposed control law was extended for an MDOF model. The mode decomposition shows that parameters for an MDOF model can be determined as in an SDOF and the non-linear influence is independent of the mode vectors.
(4) The responses of a 3-DOF structural model for seismic excitations were computed as an application example. The results showed that the NMW control forces can efficiently reduce structural responses.

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